FORCES ON FLAT INCLINED SURFACES

Need to find direction, magnitude and location of resultant force. See handout Inclined Surface for definition sketch of problem.

DIRECTION

Since pressure is a normal force. It always acts inward and perpendicular to the wetted surface.

MAGNITUDE (gage pressure formulas)

At any depth, the force acting on differential area element dA is dF=PdA=ρghdA.

Note from geometry on our picture with (0,0) at free surface h=ysinθ.

$$F_R = \int_A \rho g h dA = \int_A \rho g y \sin(\theta) dA = \rho g \sin(\theta) \int_A y dA,$$

note that $y_c = \dfrac{\int_A y dA}{A}$ is the centroid of area A

Leaving,

$$F_R = \rho g \sin(\theta) A y_c = \rho g A[\sin(\theta) y_c] = \rho g A h_c, \text{ since } h_c = y_c \sin(\theta)$$

$$F_R = \rho g h_c A$$

h_c = depth to centroid from free surface measured vertically
A = area of plane surface.

MAGNITUDE OF RESULTANT FORCE IS PRESSURE AT CENTROID TIMES AREA OF SURFACE INDEPENDENT OF ANGLE!

LOCATION

NOTE, RESULTANT DOES **NOT** ACT THROUGH CENTROID except in case of horizontal surface.

$ Resultant is located at a point called the center of pressure (x_R,y_R) or (x_p,y_p) or (x_{cp},y_{cp})

$ Must balance moments to find this point.

Start with x axis

$$Moment = F_R y_R = \int_A y dF$$

or $y_R = \dfrac{\int_A y dF}{F_R}$, substitute from before for dF=PdA and $F_R = \rho g h_c A$

$$y_R = \frac{\int_A yPdA}{\rho g h_c A}$$

$$y_R = \frac{\int_A y\rho g h dA}{\rho g h_c A} \qquad \text{since } P = \rho g h$$

$$y_R = \frac{\int_A y\rho g y \sin(\theta) dA}{\rho g y_c \sin(\theta) A} \quad \text{since } h = y\sin(\theta) \quad and \quad h_c = y_c \sin(\theta)$$

$$y_R = \frac{\rho g \sin(\theta) \int_A y^2 dA}{\rho g \sin(\theta) y_c A}$$

$$y_R = \frac{\int_A y^2 dA}{y_c A}$$

The second moment of inertia, I_x, is defined as

$I_x = \int_A y^2 dA$ = moment of inertia around an x-axis at y=0.

$\therefore y_R = \dfrac{I_x}{y_c A}$, but I_x depends on both area and depth. We have to apply parallel axis theorem so we get moment of inertia around x-axis through CENTROID.

Parallel axis theorem states

$I_x = I_{xc} + A y_c^2$, I_{xc} is the moment of inertia around x-axis but through centroid of A.

Upon substitution

$$y_R = \frac{I_{xc} + A y_c^2}{y_c A}$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

note that since $I_{xc}>0$ and $y_c A > 0$ their quotient is > 0.

So, the resultant force does not pass through centroid but always below it (except for horizontal surfaces)

Can go through same analysis for x_R to get

$x_R = \frac{I_{xyc}}{y_c A} + x_c$, where I_{xyc} is product of inertia about centroid of A.

Note $I_{xyc} = 0$ and $x_R = x_c$ is submerged area is symmetric around y-axis passing through centroid. THE CASE FOR US!

SUMMARY

The White textbook does these derivations with the coordinate axis at the location of the centroid. We have done them with the coordinate axis at the free surface. YOU MUST BE CAREFUL where you put the coordinate axis origin!

	AXIS ORIGIN AT FREE SURFACE	AXIS ORIGIN AT CENTROID (book method)
DIRECTION	Inward and normal to wetted surface	Inward and normal to wetted surface
MAGNITUDE	$F_r = \rho g h_c A$	$F_r = \rho g h_c A$
LOCATION, y_R	$y_R = \frac{I_{xc}}{y_c A} + y_c$	$y_R = -\frac{I_{xc}}{y_c A}$
LOCATION, x_R	$x_R = \frac{I_{xyc}}{y_c A} + x_c$	$x_R = -\frac{I_{xyc}}{y_c A}$