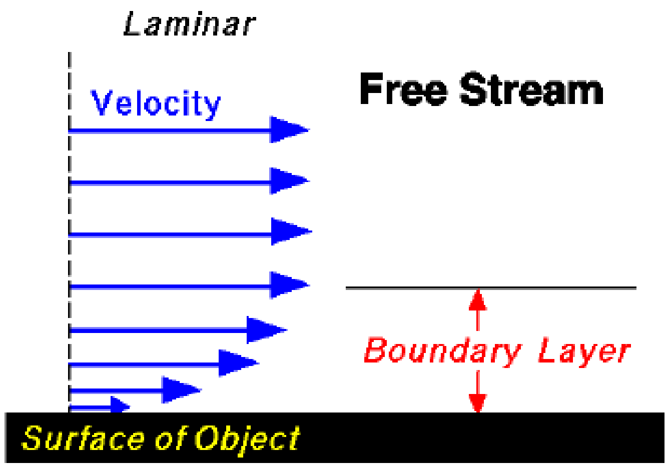
CIEG 305 HANDOUT ON EQUATION NON-DIMENSIONALIZATION

Suppose we have the case of an oscillatory flow moving back and forth across a plate (think of waves going back and forth over a sea bed). The flow near the bed is retarded by friction due to the no-slip condition.



Given an equation that describes this motion in the x-direction as:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

Find a suitable non-dimensionalization such that we can determine the order of magnitude of the terms (which are important).

We must therefore scale all variables of interest as follows (with prime indicating a non-dimensional quantity)

$x' = xk;\quad \mathrm{k} = 1/(\text{wavelength of motion})$

$z' = \frac{z}{\delta};\quad \delta$ is the boundary layer thickness

$t' = t\sigma;\quad \sigma$ is the angular frequency $= 1/\mathrm{T}$, T = wave period

$u' = \frac{u}{a\sigma};\quad \mathrm{a}$ = wave amplitude

$\mathrm{P}' = \frac{\mathrm{P}}{\rho \mathrm{ga}}$

Now we must redefine the derivatives based on the non-dimensional variables

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = k\frac{\partial}{\partial x'}$$
$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x'}\right) = \frac{\partial}{\partial x'}\left(k\frac{\partial}{\partial x'}\right)\frac{\partial x'}{\partial x} = k^2\frac{\partial^2}{\partial x'^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial}{\partial t'}\sigma = \sigma\frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z'}\frac{\partial z'}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z'}\frac{1}{\delta}\right) = \frac{\partial}{\partial z'}\left(\frac{1}{\delta}\frac{\partial}{\partial z'}\right)\frac{\partial z'}{\partial z} = \frac{1}{\delta^2}\frac{\partial^2}{\partial z'^2}$$

Now substitute into original equation

$$\frac{\partial}{\partial t'}(\sigma u' a\sigma) = -\frac{1}{\rho}k\frac{\partial}{\partial x'}(P'\rho g a) + \nu k^2\frac{\partial^2}{\partial x'^2}(u' a\sigma) + \frac{\nu}{\delta^2}\frac{\partial^2}{\partial z'^2}(u' a\sigma)$$

Collect constants outside derivatives

$$a\sigma^2\frac{\partial u'}{\partial t'} = -gak\frac{\partial P'}{\partial x'} + \nu k^2 a\sigma\frac{\partial^2 u'}{\partial x'^2} + \frac{\nu a\sigma}{\delta^2}\frac{\partial^2 u'}{\partial z'^2}$$

Divide through to isolate the time derivative

$$\frac{\partial u'}{\partial t'} = -\frac{gk}{\sigma^2}\frac{\partial P'}{\partial x'} + \frac{\nu k^2}{\sigma}\frac{\partial^2 u'}{\partial x'^2} + \frac{\nu}{\delta^2\sigma}\frac{\partial^2 u'}{\partial z'^2}$$

Now we look at constants in front of terms

For the pressure, the constant is the inverse of a Froude number which have values O(1).

For the x diffusion term the constant is the inverse of the Reynolds Number, which for typical flows is on the order of the inverse of 10^5 or 10^6. SO IT IS SMALL

Thus the constant on the vertical diffusion term must be O(1) to balance the other order 1 term. This implies

$\frac{\nu}{\delta^2\sigma} \approx 1 \quad or \quad \delta \approx \sqrt{\frac{\nu}{\sigma}}$. So we can get a feel for the expected magnitude of the boundary layer thickness.