2. Stability of a Floating Body

Objective

The objectives of this experiment are:
1. to measure the angle of inclination at which an eccentrically loaded body floats,
2. to observe the circumstances under which a floating body is unstable, and
3. to compare the observed results with theoretical predictions.

Apparatus

The apparatus consists of an open plastic box (‘barge’) which floats in water and carries a mast (Figure 1). A plumb-bob suspended from the mast provides a means of measuring the angle of inclination of the barge. The vertical position of the center of gravity is controlled by a weight $W_v$ which may be moved to different heights on the mast. The horizontal position of the center of gravity is controlled by a second weight $W_h$ which may be moved to different horizontal positions on the barge. The following information is necessary: length of barge $L = 34.9$ cm., width of barge $b = 20.3$ cm., vertically moving weight $W_v = 2.79$ N, horizontally moving weight $W_h = 3.11$ N, total weight of assembled apparatus $W = 13.21$ N. For the barge without the weights, the vertical position of the center of gravity, $z_b$, is 5.2 cm. from the outer bottom of the barge. In the following all $z$ distances are measured from the outer bottom of the barge.

Figure 1: A schematic plot of the barge and experimental apparatus
Inclination Test

The barge in the inclination test is stable and the purpose is to determine the relationship between load that brings the barge to tilt and the angle of the tilt. The theory behind the inclination test goes as follows. A floating body shall experience net vertical buoyancy force \( B \) (upward) that balance with its weight \( W \) (downward), i.e., \( B = W \). The weight of the floating body \( W \) acts through its center of gravity \((x_G, z_G)\) while the buoyancy force \( B \) acts through the centroid of the displaced volume (called buoyancy center). When the barge is inclined at an angle \( \theta \), the balance of moments about the origin \( O \) (see Figure 2) requires

\[
(x_B \cos \theta + z_B \sin \theta) - (x_G \cos \theta + z_G \sin \theta) = 0
\]  

(1)

In the above derivation, we have used the condition that the buoyant force \( B \) pushing up on the barge is equal to the total weight \( W \), as is required for a floating body. The position of the center of buoyancy can be derived from the geometry of the submerged part of the barge:

\[
x_B = \frac{b^2 \tan \theta}{12d} \quad \text{(2a)}
\]

\[
z_B = \frac{b^2 \tan^2 \theta}{24d} + \frac{d}{2} \quad \text{(2b)}
\]

where \( d \) is the submerged depth of the barge which can be evaluated as \( d = W/(\rho g L b) \), with \( \rho \) is the water density and \( g \) is the gravitational acceleration. For given \( d \) and \( b \), \( x_B \) and \( z_B \) depend on the angle \( \theta \) only.

The gravity center, with weights \( W_v \) and \( W_h \) are found from the locations of the weights

\[
x_G = \frac{W_h x_h}{W} \quad \text{(3a)}
\]

\[
z_G = \frac{W_b z_b + W_v z_v + W_h z_h}{W} \quad \text{(3b)}
\]

where \( W_b = \) the barge weight given by \( W_b = (W - W_v - W_h) \). \((x_b, z_b)\), \((x_v, z_v)\) and \((x_h, z_h)\) are respectively the coordinates for the location of the center of gravity of the horizontal weight, the vertical weight, and the barge. In Equation (3a), we have specified \( x_b = 0 \) and \( x_v = 0 \) appropriate for this experiment. The values of \( z_v \), \( x_h \) and \( z_h \) are determined by the locations of the weights, and \( z_b = 5.2 \) cm as stated above. Substitution of Equation (2) into Equation (1) yields the following 3rd order algebraic equation for \( \tan \theta \):

\[
\frac{b^2}{24d} \tan^3 \theta + \left( \frac{b^2}{12d} + \frac{d}{2} - z_G \right) \tan \theta - x_G = 0
\]  

(4)
Figure 2: Definition of the coordinate system, $\theta$, $(x_G, z_G)$ and $(x_B, z_B)$.

**Stability Test**

This test aims to verify the stability criterion that can be derived from the theory above. The stability criterion for the barge may be described as follows. First, with the barge in horizontal position, a small tilt of $\theta$ is provided. The weight of the floating body provides a rotating moment and tend to destabilize the system. On the other hand, the buoyancy force provides a counter moment that tends to stabilize the system. If Eq. (1) is positive for a small value of $\theta$, then the buoyancy force acting on the barge due to the tilt can bring the barge back to an upright position, implying the barge is stable. However, if $z_G$ becomes too large, the gravity force acting on the barge rotates the barge from its upright position for any small $\theta$ and the buoyancy force cannot restore the stability and hence the barge is unstable. $z_{Gcrit}$ is the value of $z_G$ corresponding to the neutral stability for small $\theta$. Show that the critical position of $z_G$ for the special case of $x_G = 0$ is given by
Eq. (5) was derived by using the conditions $x_G = 0$ and $\tan \theta << 1$ in Equation (4). If the actual $z_G$ is above $z_{G\text{crit}}$, the barge is unstable since the higher $z_G$ produces an overturning moment that the buoyancy force cannot conquer. On the other hand, if $z_G < z_{G\text{crit}}$, the restoring moment is found and the system is stable.

**Procedure**

You are requested to perform an inclination test and a stability test. The inclination test verifies Eq. (4) and the stability test verifies Eq. (5). The experimental procedure is the following:

1. For the inclination test, each person in the lab group should set up one combination of vertical and horizontal weight positions (i.e., the values of $z_v$ and $x_h$), and measure the resulting angle of inclination $\theta$ of the barge.
2. For the stability test, the horizontal weight $W_h$ can be moved to the mast of the barge. Thus $x_G = 0$ in Eq. (5) in this experiment. By trial and error, find the minimum vertical position of the weights at which the apparatus becomes unstable. Record the vertical position of both weights and calculate the equivalent value of $z_{G\text{crit}}$, the center of gravity for the entire barge. Compare this with the theoretical value given by Eq. (5). This procedure is somewhat time consuming due to the sensitivity of the floating body close to the stability limit. Try to get more than one independent measurement. Start over each time with weights located well below the stable limit, raising them in small incremental steps each time checking the stability. [Note: A stable barge has a rolling period. As the weights on the mast are moved upward the rolling period increases until the barge becomes unstable at the critical position.]
3. Record the water temperature to determine the density $\rho$.

**Data Analysis and Results**

In the various sections of your report, it should include the following:

1. In the theoretical background section fill in all the missing steps in the derivations of Equations (4) and (5). If you can also derive (2a,b), (3a,b), extra credits (4 points) will be given.
2. In the inclination test, compute the solution to Eq. (4) for all the combinations of $x_h$ and $z_v$ used in the experiment. Describe how you solve Eq. (4) (see also Appendix).
3. Then plot the measured and computed values of $\theta$ obtained from Equation (4).
4. What is the correlation coefficient $r$ between the theoretical and measured values? Discuss the following questions: Does the value for $r$ indicate your experiments were accurate and reliable? What do you think are the most important error sources? Can you suggest improvements to the inclination experiment that could reduce the error?
5. In the stability test, compare the critical position $z_{G,\text{crit}}$ when the barge starts to become unstable with the theoretical value given by Eq. (5). Is the agreement satisfactory? What do you think are the most important error sources? Can you suggest improvements to the stability experiment that could reduce the error?

Appendix

The roots of 3$^{rd}$ order algebraic equations, such as Equation (4), can be found by many different approaches/tools. Here, we suggest three options:

(1) Check out a mathematical handbook. You shall be able to find the solution for the roots of 3$^{rd}$-order polynomial.

(2) Using Matlab. There is a Matlab command “roots”. This commend can find all the roots for a given N-order polynomial.

(3) Write your own Newton-Raphson numerical scheme to solve for the root.